

# spherically symmetric potential

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$$\left[ -\frac{\hbar^2}{2m} \Delta_r + V(|\vec{r}|) \right] \psi(\vec{r}) = E \psi(\vec{r})$$

separation ansatz

$$\psi(\vec{r}) = \frac{u(r)}{r} Y_{l,m}(\theta, \phi)$$

$$-i \frac{\partial}{\partial \vartheta} Y_{l,m}(\vartheta, \varphi) = m Y_{l,m}(\vartheta, \varphi)$$

$$- \left( \frac{1}{\sin(\vartheta)} \frac{\partial}{\partial \vartheta} \sin(\vartheta) \frac{\partial}{\partial \vartheta} + \frac{1}{\sin^2(\vartheta)} \frac{\partial^2}{\partial \varphi^2} \right) Y_{l,m}(\vartheta, \varphi) = l(l+1) Y_{l,m}(\vartheta, \varphi)$$

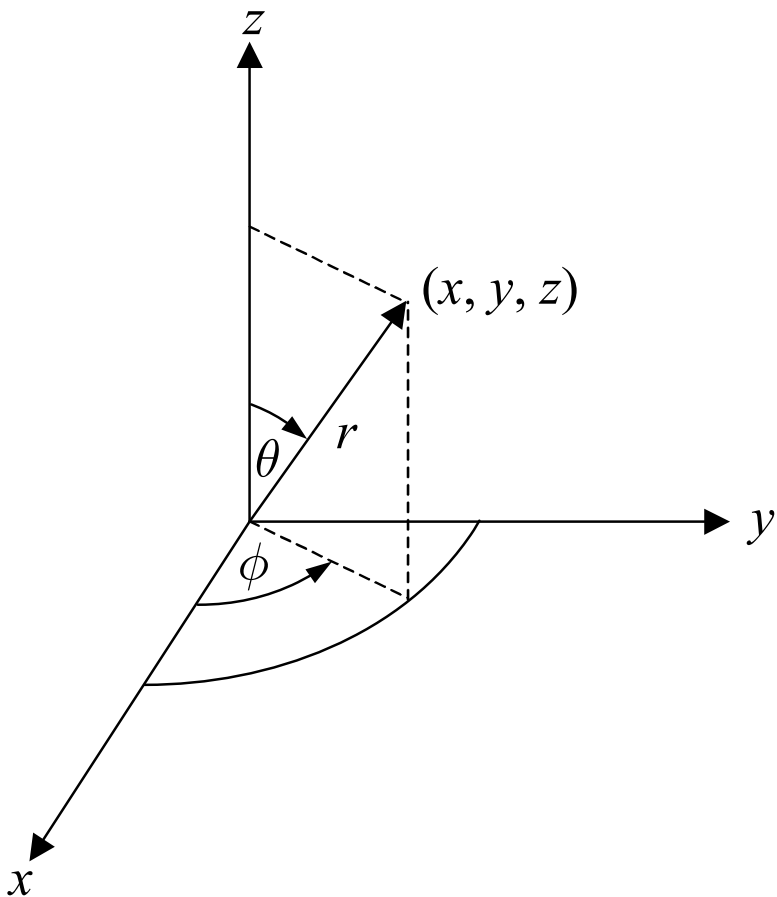
$$\left( -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} - V(r) \right) u(r) = E u(r)$$

# Spherical Harmonics

spherical harmonics (Condon-Shortley phase-convention)

$$Y_{l,m}(\theta, \phi) = (-1)^{\frac{m+|m|}{2}} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-|m|)!}{(l+|m|)!}} P_{l,m}(\cos \theta) e^{im\phi}$$

where  $P_{l,m}(\cos \theta) = \frac{(-1)^l}{2^l l!} \sin^m \theta \frac{d^{l+m} \sin^{2l} \theta}{d \cos^{l+m} \theta}$  associated Legendre functions



properties

$$\int d\Omega \overline{Y_{l,m}(\theta, \phi)} Y_{l',m'}(\theta, \phi) = \delta_{l,l'} \delta_{m,m'} \quad \text{orthonormality}$$

$$\sum_{l=0}^{\infty} \sum_{m=-l}^l Y_{l,m}(\theta, \phi) \overline{Y_{l,m}(\theta', \phi')} = \frac{\delta(\theta - \theta') \delta(\phi - \phi')}{\sin \theta} \quad \text{completeness}$$

$$Y_{l,m}(\pi - \theta, \pi + \phi) = (-1)^l Y_{l,m}(\theta, \phi) \quad \text{parity}$$

$$Y_{l,-m}(\theta, \phi) = (-1)^m \overline{Y_{l,m}(\theta, \phi)}$$

# angular momentum algebra

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$$\vec{J} = \begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} \text{ where } J_i^\dagger = J_i \text{ and } [J_1, J_2] = iJ_3 \text{ (cyclic)}$$

orthonormal eigensystem

$$\begin{aligned} \vec{J}^2 |j, m\rangle &= j(j+1) |j, m\rangle & j \in \{0, 1/2, 1, 3/2, 2, \dots\} \\ J_3 |j, m\rangle &= m |j, m\rangle & m \in \{-j, -j+1, \dots, j\} \end{aligned}$$

ladder operators  $J_\pm = J_1 \pm iJ_2$

$$J_\pm |j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)} |j, m \pm 1\rangle$$

Condon-Shortley phase convention

# angular momentum states

$$\begin{aligned}\vec{J}^2 |j, m\rangle &= j(j+1) |j, m\rangle \\ J_3 |j, m\rangle &= m |j, m\rangle\end{aligned}$$

